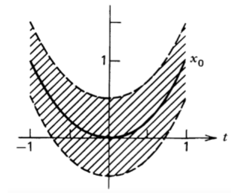
Section 1.3, page 23 # 2, 6, 8, 9, 14

2. What is an open ball (a) $B(x_0; 1)$ on $\mathbb{R}$? (b) In $\mathbb{C}$? (Cf. 1.1-5) (c) In $C[a, b]$? (Cf. 1.1-7) (d) Explain Fig. 8.



(a) On $\mathbb{R}$, $B(x_0; 1) = (-1, 1) \subset \mathbb{R}$.
(b) In $\mathbb{C}$, $B(x_0; 1)$ is a disc of radius 1, centered at x_0.
(c) In $C[a, b]$, $B(x_0; 1)$ is a strip of width 1, about the function x_0.
(d) In Fig. 8 illustrates $B(x_0; 1/2)$.

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6. If x_0 is an accumulation point of a set $A \subset (X,d)$, show that any neighborhood of x_0 contains infinitely many points of A.

Proof:
Since x_0 is an accumulation point of X, then $\exists\, x_1 \in B(x_0, 1)^* \cap A$, where $* = \setminus\{x_0\}$.
Then $\exists\, x_2 \in B(x_0, 1/2)^*$.
Let $n > 1$ and assume $\exists\, x_n \in B(x_0, 1/n)^* \cap A$.
Then $\exists\, x_{n+1} \in B(x_0, 1/(n+1)) \cap A$.
Thus, by induction, we have that infinitely many points of A lie in $B(x_0, 1)$.

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8. Show that the closure $\overline{B(x_0;r)}$ of an open ball $B(x_0;r)$ in a metric space can differ from the closed ball $\tilde{B}(x_0;r)$.

Proof:
Let $X = \mathbb{Z}$. Then $\{-1, 0, 1\} = \tilde{B}(0; 1) \neq \overline{B(0; 1)} = B(0; 1) = \{0\}$.

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9. Show that (a) $A \subset \overline{A}$, (b) $\overline{\overline{A}} = \overline{A}$, (c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, (d) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.
Proof:
(a) Let $A' = \{x \in \overline{A} \mid x$ is an accumulation point of $A\}$. Then $A \subset A \cup A' = \overline{A}$.
(b) $\overline{A} \subset \overline{A} \cup \left(\overline{A}\right)' = \overline{\overline{A}}$.

For the reverse inclusion, let $x \in \overline{\overline{A}}$ and note that $x \in \overline{A}$ or $x \in \left(\overline{A}\right)'$.
If $x \in \overline{A}$, we are done. Suppose $x \notin \overline{A}$.
Then $x \in \left(\overline{A}\right)'$ and $x \in \left(\overline{A}\right)^c$ which is open. Thus $\exists\, r > 0 \ni x \in B(x, r) \subset \left(\overline{A}\right)^c$.
But $x \in \left(\overline{A}\right)' \Rightarrow \exists\, y \in \overline{A}$ such that $y \in B(x, r)$, a contradiction to $B(x, r) \subset \left(\overline{A}\right)^c$.
$\therefore\ \overline{\overline{A}} \subset \overline{A}$.

9. (c) Let $\varepsilon > 0$. Then

$x \in (A \cup B)' \Leftrightarrow \exists\, y \in B(x, \varepsilon)^* \cap (A \cup B) = [B(x, \varepsilon)^* \cap A] \cup [B(x, \varepsilon)^* \cap B]$

$\Leftrightarrow y \in B(x, \varepsilon)^* \cap A$ or $y \in B(x, \varepsilon)^* \cap B$

$\Leftrightarrow x \in A'$ or $x \in B'$

$\Leftrightarrow x \in A' \cup B'$

$\Leftrightarrow (A \cup B)' = A' \cup B'$.

This gives us that

$\overline{A \cup B} = A \cup B \cup (A \cup B)' = A \cup B \cup A' \cup B' = A \cup A' \cup B \cup B' = \overline{A} \cup \overline{B}$.

(d) Let $\varepsilon > 0$. Let $x \in (A \cap B)'$.

Then $\exists\, y \in B(x, \varepsilon)^* \cap (A \cap B) = [B(x, \varepsilon)^* \cap A] \cap [B(x, \varepsilon)^* \cap B]$.

Thus, $y \in B(x, \varepsilon)^* \cap A$ and $y \in B(x, \varepsilon)^* \cap B$.

Hence $x \in A'$ and $x \in B'$, or equivalently, $x \in A' \cup B'$. So then $(A \cup B)' \subset A' \cup B'$.

This gives us that

$\overline{A \cap B} = A \cap B \cup (A \cap B)' \subset A \cap B \cup (A' \cap B') = (A \cup A') \cap (B \cup B') = \overline{A} \cap \overline{B}$.

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14. Show that a mapping $T{:}X \to Y$ is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X.

Proof:

$\Rightarrow$: Suppose T is continuous.

Let $V \subset Y$ be closed.

Then V^c is open, hence by continuity of T and theorem 1.3-4, $T^{-1}(V^c)$ is open.

Thus, $T^{-1}(V) = [T^{-1}(V^c)]^c$ is closed.

$\Leftarrow$: Let $V \subset Y$ be open, then V^c is closed.

And, by assumption, $T^{-1}(V^c) = [T^{-1}(V)]^c$ is closed. This gives us that $T^{-1}(V)$ is open.

So then T is continuous by theorem 1.3-4.
