Section 1.4, page 31 # 1, 2, 3, 4, 5

1. If a sequence (x_n) in a metric space X is convergent and has limit x, show that every subsequence (x_{nk}) of (x_n) is convergent and has the same limit x.

Proof:

Let $\varepsilon > 0$. Let $(x_n) \subset X$ such that $x_n \to x$. Then $\exists\, N \in \mathbb{N} \ni d(x_n, x) < \varepsilon\ \forall\, n \geq N$.

Let $(x_{nk}) \subset (x_n)$. $\exists\, k_0 \in \mathbb{N} \ni n_{k_0} > N\ \forall\, k \geq k_0$.

$\therefore$ $d(x_{nk}, x) < \varepsilon\ \forall\, k \geq k_0$. Hence $x_{nk} \to x$.

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2. If (x_n) is Cauchy and has a convergent subsequence, say, $x_{n_k} \to x$, show that (x_n) is convergent with the limit x.

Proof:

Let $\varepsilon > 0$. Let $(x_n) \subset X$ such that (x_n) is Cauchy.

Then $\exists\, N \in \mathbb{N} \ni \forall\, m, n \geq N$, $d(x_m, x_n) < \varepsilon/2$.

Suppose $\exists\, (x_{nk}) \subset (x_n) \ni x_{nk} \to x$. Then $\exists\, k_0 \in \mathbb{N} \ni n_{k_0} > N\ \forall\, k \geq k_0$ and $d(x_{nk}, x) < \varepsilon/2$.

So then $\forall\, k \geq k_0$ $d(x_n, x) \leq d(x_n, x_{nk}) + d(x_{nk}, x) < \varepsilon$, hence $x_n \to x$.

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3. Show that $x_n \to x \Leftrightarrow$ for every neighborhood V of x there is an integer n_0 such that $x_n \in V$ for all $n > n_0$.

Proof:

$\Rightarrow$: Assume $x_n \to x$ and let V be a neighborhood of x. Then $\exists\, \varepsilon > 0 \ni B(x, \varepsilon) \subset V$.

And $x_n \to x \Rightarrow \exists\, n_0 \in \mathbb{N} \ni d(x_n, x) < \varepsilon\ \forall\, n \geq n_0$. Thus $x_n \in B(x, \varepsilon) \subset V\ \forall\, n \geq n_0$.

$\Leftarrow$: Let $\varepsilon > 0$. Then, by assumption, $\exists\, n_0 \in \mathbb{N} \ni \forall\, n > n_0$, $x_n \in B(x, \varepsilon)$.

Thus, $\forall\, n > n_0$, $d(x_n, x) < \varepsilon$, hence $x_n \to x$.

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4. Show that a Cauchy sequence is bounded.

Proof:

Let $(x_n) \subset X$ such that (x_n) is Cauchy.

Then $\exists\, N \in \mathbb{N} \ni \forall\, m, n \geq N$, $d(x_m, x_n) < 1$. Thus, $\forall\, n \geq N$, $x_n \in B(x_N, 1)$.

Let $r = \max(d(x_1, x_N), d(x_2, x_N), \ldots, d(x_{N-1}, x_N), 1)$.

Then $\forall\, n \in \mathbb{N}$ $x_n \in B(x_N, r)$.

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5. Is boundedness of a sequence in a metric space sufficient for the sequence to be Cauchy? No. Convergent? No.

Proof:

Let $X = \mathbb{Z}$ and let $x_n = \begin{cases} 0 : n \text{ is even} \\ 1 : n \text{ is odd} \end{cases}$.

This sequence is bounded by 1, but $\forall\, n \in \mathbb{N}$, $d(x_n, x_{n+1}) = 1$, hence is not Cauchy.

Additionally, the sequence is not convergent as it contains 2 subsequences that converge to different limits. This contradicts the contrapositive of exercise #1.
