Section 1.5, page 39 # 5, 6, 7, 8

5. Show that the set X of all integers with metric d defined by $d(m, n) = |m - n|$ is a complete metric space.

Proof:
We know $(\mathbb{R}, d)$ is complete and $X \subset \mathbb{R}$, so we only need to show X is closed under d.
Let $(x_n) \subset X$ such that $x_n \to x$ for some $x \in \mathbb{R}$.
Let $\varepsilon > 0 \ni \varepsilon < 1/2$. Then $\exists\, N \in \mathbb{N} \ni \forall\, n \geq N$, $|x_n - x| < \varepsilon$.
Thus, $\forall\, m, n \geq N$, $x_n = x_m$, as distinct integers are at least a distance of 1 from each other. This means the sequence is eventually constant, which gives us that $x = x_N \in X$, hence X is closed.

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6. Show that the set of all real numbers constitutes an incomplete metric space if we choose $d(x, y) = |\text{arc tan } x - \text{arc tan } y|$.

Proof:
Let $x_n = n$. Let $\varepsilon > 0$.
Choose $\exists\, N \in \mathbb{N} \ni N > \tan(\pi/2 - \varepsilon/2)$. Then $\pi/2 - \text{arc tan } N < \varepsilon/2$.
This gives us that $\forall\, n \geq N$,

$$\begin{aligned} & |\text{arc tan } x_n - \text{arc tan } x_{n+1}| \\ = \; & |\text{arc tan } n - \text{arc tan } (n+1)| \\ = \; & |\text{arc tan } n - \pi/2 + \pi/2 - \text{arc tan } (n+1)| \\ \leq \; & |\text{arc tan } n - \pi/2| + |\pi/2 - \text{arc tan } (n+1)| \\ < \; & \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus (x_n) is Cauchy. However, (x_n) does not converge.
For if $\exists\, x \in \mathbb{R} \ni x_n \to x$, then x must be the least upper bound of (x_n) as x_n is increasing. However, we can find $n \in \mathbb{N}$ such that $n = x_n > x$, a contradiction.

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7. Let X be the set of all positive integers and $d(m, n) = |m^{-1} - n^{-1}|$. Show that (X, d) is not complete.

Proof:
Let $x_n = n$. Let $\varepsilon > 0$.
Choose $N \in \mathbb{N} \ni N > 1/\varepsilon$. Then $\forall\, n \geq N$, $|x_n - x_{n+1}| = |1/n - 1/(n+1)| < 1/n < \varepsilon$.
Thus (x_n) is Cauchy. However, (x_n) does not converge.
For if $\exists\, x \in \mathbb{R} \ni x_n \to x$, then x must be the least upper bound of (x_n) as x_n is increasing. However, we can find $n \in \mathbb{N}$ such that $n = x_n > x$, a contradiction.

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8. Show that the subspace $Y \subset \mathscr{C}[a, b]$ consisting of all $x \in \mathscr{C}[a, b]$ such that $x(a) = x(b)$ is complete.

Proof:
We know $(\mathscr{C}[a, b], d_\infty)$ is complete. So then we only need to show Y is closed.
Let $\varepsilon > 0$. Let $(f_n) \subset Y$. Then $f_n \to f$ where $f \in \mathscr{C}[a, b]$.
Thus $\exists\, N \in \mathbb{N} \ni \forall\, n \geq N$, $\sup_{a \leq x \leq b} |f_n(x) - f(x)| < \varepsilon/2$. So now we have

$$\begin{aligned}
|f(a) - f(b)| &\leq |f_n(a) - f(a)| + |f_n(a) - f_n(b)| + |f_n(b) - f(b)| \\
&\leq \sup_{a \leq x \leq b} |f_n(a) - f(a)| + |f_n(a) - f_n(b)| + \sup_{a \leq x \leq b} |f_n(b) - f(b)| \\
&< \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

$\therefore f(a) = f(b)$. Since $f \in \mathscr{C}[a, b]$, then $f \in Y$.