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4. Which of the following subsets of $\mathbb{R}^3$ constitute a subspace of $\mathbb{R}^3$? [Here, $x = (\xi_1, \xi_2, \xi_3)$.]
(a) All x with $\xi_1 = \xi_2$ and $\xi_3 = 0$.
(b) All x with $\xi_1 = \xi_2 + 1$.
(c) All x with positive ξ_1, ξ_2, ξ_3.
(d) All x with $\xi_1 - \xi_2 + \xi_3 = k$ = constant.
Proof:
(a) Let $A = \{x \in \mathbb{R}^3 | \ x = (\xi_1, \xi_2, \xi_3)$ with $\xi_1 = \xi_2$ and $\xi_3 = 0\}$.
$(0, 0, 0) \in A$, so $A \neq \varnothing$.
Let $x = (a_1, a_2, 0), y = (b_1, b_2, 0) \in A$.
Then $(a_1, a_2, 0) + (b_1, b_2, 0) = (a_1 + b_1, a_2 + b_2, 0)$ and $a_1 + b_1 = a_2 + b_2$, so $x + y \in A$.
Let $c \in \mathbb{R}$, then $c(a_1, a_2, 0) = (ca_1, ca_2, 0)$ and $ca_1 = ca_2$, so $cx \in A$.
$\therefore$ A is a subspace of $\mathbb{R}^3$.
(b) Let $B = \{x \in \mathbb{R}^3 | \ x = (\xi_1, \xi_2, \xi_3)$ with $\xi_1 = \xi_2 + 1\}$.
Since $0 \neq 0 + 1$, then $(0, 0, 0) \notin B$. $\therefore$ B is not a subspace of $\mathbb{R}^3$.
(c) Let $C = \{x \in \mathbb{R}^3 | \ x = (\xi_1, \xi_2, \xi_3)$ with all positive $\xi_1, \xi_2, \xi_3\}$.
Since 0 is not a positive number, then $(0, 0, 0) \notin C$. $\therefore$ C is not a subspace of $\mathbb{R}^3$.
(d) Let $D = \{x \in \mathbb{R}^3 | \ x = (\xi_1, \xi_2, \xi_3)$ with $\xi_1 - \xi_2 + \xi_3 = k$ = constant$\}$.
Let $x = (a_1, a_2, a_3), y = (b_1, b_2, b_3)$.
Then $x + y = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ and $a_1 + b_1 - a_2 - b_2 + a_3 + b_3 = 2k$.
So $x + y \notin D$, hence D is not a subspace of $\mathbb{R}^3$ unless $D = \{0\}$.

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5. Show that $\{x_1, x_2, ..., x_n\}$, where $x_j(t) = t^j$, is a linearly independent set in the space $C[a, b]$.

Proof:
Let $\alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n = 0 \ \forall \ t \in [a, b]$, where $\alpha_1, \alpha_2, ..., \alpha_n \in K$, and $\alpha_1, \alpha_2, ..., \alpha_n$ are not all zero.
By the fundamental theorem of algebra, there are at most n values of t that make this equation true. However, there are infinitely many values $t \in [a, b]$. Thus the only possible choice for $\alpha_1, \alpha_2, ..., \alpha_n \in K$ is $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$.

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6. Show that in an n-dimensional vector space X, the representation of any x as a linear combination of given basis vectors $e_1, e_2, ..., e_n$ is unique.

Proof:
Let X have dimension n. Suppose $\exists \ \alpha_1, \alpha_2, ..., \alpha_n \in K$, and $\beta_1, \beta_2, ..., \beta_n \in K$, such that $\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n$. Then
$(\alpha_1 - \beta_1)e_1 + (\alpha_2 - \beta_2)e_2 + \cdots + (\alpha_n - \beta_n)e_n = 0$. Since X has dimenstion n, then $\{e_1, e_2, ..., e_n\}$ is linearly independent. Thus $\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \cdots = \alpha_n - \beta_n = 0$, hence $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \cdots, \alpha_n = \beta_n$. $\therefore$ $\forall \ \alpha_1, \alpha_2, ..., \alpha_n \in K$, $\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$ is unique.

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10. If Y and Z are subspaces of a vector space X, show that $Y \cap Z$ is a subspace of X, but $Y \cup Z$ need not be one. Give examples.

Proof:
Let $v, w \in Y \cap Z$. Then $v, w \in Y$ and $v, w \in Z$. Hence $\alpha v + w \in Y \cap Z \; \forall \; \alpha \in K$.
However, if $X = \mathbb{R}^3$, $Y = \{(a, 0, 0) | \; a \in \mathbb{R}\}$ and $Z = \{(0, b, 0) | \; b \in \mathbb{R}\}$, then $(1, 0, 0) \in Y$ and $(0, 1, 0) \in Z$, but $(1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin Y \cup Z$. $\therefore$ $Y \cup Z$ is not a subspace of $\mathbb{R}^3$.