Section 2.2, page 64 # 3, 4, 6, 8, 9, 11, 13

3. Prove (2), $|\,\|y\| - \|x\|\,| \le \|y - x\|$ where $x, y \in X$, a normed space.
Proof:

$\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$.
$\|y\| = \|y - x + x\| \le \|y - x\| + \|x\|$.
Thus, $|\,\|x\| - \|y\|\,| \le \|x - y\|$.

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4. (a) Show that we may replace (N2), $\|x\| = 0 \Leftrightarrow x = 0$, by $\|x\| = 0 \Rightarrow x = 0$ without altering the concept of a norm. (b) Show that nonegativity of a norm also follows from (N3) and (N4).

Proof:
(a) If $x = \Theta$, then $x = 0x$. So then $\|x\| = \|0x\| = |0|\,\|x\| = 0$, hence $x = \Theta \Rightarrow \|x\| = 0$. Thus the reverse implication of (N2) follows from (N3).
(b) $0 = \|x\| - \|x\| = \|x + x - x\| - \|x\| \le \|x\| + \|x\| + \|x\| - \|x\| = 2\|x\| \Rightarrow 0 \le \|x\|$. So then $\|x\| \ge 0$ follows from (N4).

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6. Let X be the vector space of all ordered pairs $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$, ... of real numbers. Show that norms of X are defined by
(a) $\|x\|_1 = |\xi_1| + |\xi_2|$
(b) $\|x\|_2 = (\xi_1^2 + \xi_2^2)^{1/2}$
(c) $\|x\|_\infty = \max\{|\xi_1|, |\xi_2|\}$
Proof:
(a) (N1) $\|x\|_1 = |\xi_1| + |\xi_2| \ge 0$ is evident.
(N2) $|\xi_1| + |\xi_2| = 0 \Rightarrow |\xi_1| = 0 = |\xi_2|$. So then $x = (0, 0) = \Theta$.
(N3) Let $\alpha \in K$.
Then $\|\alpha x\|_1 = |\alpha\xi_1| + |\alpha\xi_2| = |\alpha||\xi_1| + |\alpha||\xi_2| = |\alpha|(|\xi_1| + |\xi_2|) = |\alpha|\|x\|_1$.
(N4) Let $x, y \in X$. Then
$\|x + y\|_1 = |\xi_1 + \eta_1| + |\xi_2 + \eta_2| \le |\xi_1| + |\xi_2| + |\eta_1| + |\eta_2| = \|x\|_1 + \|y\|_1$.
(b) (N1) $\|x\|_2 = (\xi_1^2 + \xi_2^2)^{1/2} \ge 0$ is evident.
(N2) $(\xi_1^2 + \xi_2^2)^{1/2} = 0 \Rightarrow \xi_1^2 = 0 = \xi_2^2$. So then $\xi_1 = 0 = \xi_2$, hence $x = (0, 0) = \Theta$.
(N3) Let $\alpha \in K$.
Then $\|\alpha x\|_2 = ((\alpha\xi_1)^2 + (\alpha\xi_2)^2)^{1/2} = (\alpha^2(\xi_1^2 + \xi_2^2))^{1/2} = |\alpha|(\xi_1^2 + \xi_2^2)^{1/2} = |\alpha|\|x\|_2$.
(N4) Let $x, y \in X$. Then, since Minkowski gives us that
$\left(\sum_{i=1}^{n}|x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}$, then for p = 2, we have
$\|x + y\|_2 = [(\xi_1 + \eta_1)^2 + (\xi_2 + \eta_2)^2]^{1/2} \le (\xi_1^2 + \xi_2^2)^{1/2} + (\eta_1^2 + \eta_2^2)^{1/2} = \|x\|_2 + \|y\|_2$.

6. (c) (N1) $||x||_\infty = \max\{|\xi_1|, |\xi_2|\} \geq 0$ is evident.
(N2) $\max\{|\xi_1|, |\xi_2|\} = 0 \Rightarrow |\xi_1| = 0 = |\xi_2|$. So then $x = (0, 0) = \Theta$.
(N3) Let $\alpha \in K$.
Then $||\alpha x||_\infty = \max\{|\alpha\xi_1|, |\alpha\xi_2|\} = \max\{|\alpha||\xi_1|, |\alpha||\xi_2|\} = |\alpha|||x||_1$.
(N4) Let $x, y \in X$. Then
$||x + y||_\infty = \max\{|\xi_1 + \eta_1|, |\xi_2 + \eta_2|\}$
$\leq \max\{|\xi_1| + |\eta_1|, |\xi_2| + |\eta_2|\}$
$\leq \max\{|\xi_1|, |\xi_2|\} + \max\{|\eta_1|, |\eta_2|\} = ||x||_\infty + ||y||_\infty$.

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8. There are several norms of practical importance on the vector space of ordered n-tuples of number (cf. 2.2-2) notably those defined by
(a) $||x||_1 = |\xi_1| + |\xi_2| + \cdots + |\xi_n|$
(b) $||x||_p = (|\xi_1|^p + |\xi_2|^p + \cdots + |\xi_n|^p)^{1/p}$
(c) $||x||_\infty = \max\{|\xi_1|, ..., |\xi_n|\}$.
In each case, verify that (N1) to (N4) are satisfied.
Proof:
(a) (N1) $||x||_1 = |\xi_1| + |\xi_2| + \cdots + |\xi_n| \geq 0$ is evident.
(N2) $|\xi_1| + |\xi_2| + \cdots + |\xi_n| = 0 \Rightarrow |\xi_1| = |\xi_2| = \cdots = |\xi_n| = 0$. So then $x = (0, 0, ..., 0) = \Theta$.
(N3) Let $\alpha \in K$.
Then $||\alpha x||_1 = |\alpha\xi_1| + |\alpha\xi_2| + \cdots + |\alpha\xi_n| = |\alpha||\xi_1| + |\alpha||\xi_2| + \cdots + |\alpha||\xi_n| =$
$|\alpha|(|\xi_1| + |\xi_2| + \cdots + |\xi_n|) = |\alpha|||x||_1$.
(N4) Let $x, y \in X$. Then $||x + y||_1 = |\xi_1 + \eta_1| + |\xi_2 + \eta_2| + \cdots + |\xi_n + \eta_n|$
$\leq |\xi_1| + |\xi_2| + \cdots + |\xi_n| + |\eta_1| + |\eta_2| + \cdots + |\eta_n|$
$= ||x||_1 + ||y||_1$.
(b) (N1) $||x||_p = (|\xi_1|^p + |\xi_2|^p + \cdots + |\xi_n|^p)^{1/p} \geq 0$ is evident.
(N2) $(|\xi_1|^p + |\xi_2|^p + \cdots + |\xi_n|^p)^{1/p} = 0 \Rightarrow \xi_1^p = \xi_2^p = \cdots = \xi_n^p = 0$ So then
$\xi_1 = \xi_2 = \cdots = \xi_n = 0$, hence $x = (0, 0, ..., 0) = \Theta$.
(N3) Let $\alpha \in K$.
Then $||\alpha x||_p = (|\alpha\xi_1|^p + |\alpha\xi_2|^p + \cdots + |\alpha\xi_n|^p)^{1/p} = (|\alpha^p||\xi_1|^p + |\alpha^p||\xi_2|^p + \cdots + |\alpha^p||\xi_n|^p)^{1/p}$
$= |\alpha|(|\xi_1|^p + |\xi_2|^p + \cdots + |\xi_n|^p)^{1/p} = |\alpha|||x||_p$.
(N4) Let $x, y \in X$. Then, using Minkowski's inequality, we have
$$||x + y||_p = \left(\sum_{i=1}^{n}|x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p} = ||x||_p + ||y||_p.$$
(c) (N1) $||x||_\infty = \max\{|\xi_1|, ..., |\xi_n|\} \geq 0$ is evident.
(N2) $\max\{|\xi_1|, ..., |\xi_n|\} = 0 \Rightarrow |\xi_1| = |\xi_2| = ... = |\xi_n| = 0$. So then $x = (0, 0, ..., 0) = \Theta$.
(N3) Let $\alpha \in K$.
Then $||\alpha x||_\infty = \max\{|\alpha\xi_1|, |\alpha\xi_2|, ..., |\alpha\xi_n|\} = \max\{|\alpha||\xi_1|, |\alpha||\xi_2|, ..., |\alpha||\xi_n|\} = |\alpha|||x||_1$.
(N4) Let $x, y \in X$. Then
$||x + y||_\infty = \max\{|\xi_1 + \eta_1|, |\xi_2 + \eta_2|, ..., |\xi_n + \eta_n|\}$
$\leq \max\{|\xi_1| + |\eta_1|, |\xi_2| + |\eta_2|, ..., |\xi_n| + |\eta_n|\}$
$\leq \max\{|\xi_1|, |\xi_2|, ..., |\xi_n|\} + \max\{|\eta_1|, |\eta_2|, ..., |\eta_n|\} = ||x||_\infty + ||y||_\infty$.

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9. Verify that (5), $\| x \| = \max_{t \in [a,b]} |x(t)|$, defines a norm.

Proof:
(N1) $\|x\| = \max_{t \in [a,b]} |x(t)| \geq 0$ is evident.
(N2) $\max_{t \in [a,b]} |x(t)| = 0 \Rightarrow x(t) = 0$ for all t. So then $x = \Theta$.
(N3) Let $\alpha \in K$.
Then $\|\alpha x\| = |\alpha| \max |x(t)| = |\alpha| \|x\|$.
(N4) Let $x, y \in X$. Then
$\|x + y\|_1 = |\xi_1 + \eta_1| + |\xi_2 + \eta_2| \leq |\xi_1| + |\xi_2| + |\eta_1| + |\eta_2| = \|x\|_1 + \|y\|_1$.

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11. A subset A of a vector space X is said to be convex if $x, y \in A$ implies
$M = \{z \in X \mid z = \alpha x + (1 - \alpha)y, 0 \leq \alpha \leq 1\} \subset A$.
M is called a closed segment with boundary points x and y; any other $z \in M$ is called an interior point of M. Show that the closed unit ball $\tilde{B}(0;1) = \{x \in X \mid \|x\| \leq 1\}$ in a normed space X is convex.

Proof:
Let $x, y \in \tilde{B}(0;1)$. Let $\alpha \in [0, 1]$. Let $z = \alpha x + (1 - \alpha)y$. Then

$$
\begin{aligned}
\| z - \Theta \| &= \| \alpha x + (1 - \alpha)y - \Theta \| \\
&= \| \alpha x + (1 - \alpha)y \| \\
&\leq \| \alpha x \| + \|(1 - \alpha)y \| \\
&= |\alpha| \, \| x \| + |(1 - \alpha)| \, \| y \| \\
&\leq \alpha \bullet 1 + (1 - \alpha) \bullet 1 = 1.
\end{aligned}
$$

$\therefore z \in \tilde{B}(0;1)$.

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13. Show that the discrete metric on a vector space $X \neq \{0\}$ cannot be obtained from a norm (Cf. 1.1-8).

Proof:
Suppose (X, d) is a vector space with the discrete metric.
If X is a normed space, then for $\forall\, x \neq y \in X$, $\alpha \neq 0 \in K$, $\alpha x \neq \alpha y$. So then $\|\alpha x - \alpha y\| = 1$,
However, by metric and norm properties,
$\|\alpha x - \alpha y\| = \|\alpha(x - y)\| \leq |\alpha| \|x - y\| = |\alpha| \bullet 1 = \alpha$, a contradiction, clearly.
