Section 2.3, page 70 # 2, 3, 5, 6, 10, 11

2. Show that c_0, the space of all sequences of scalars converging to zero, is a closed subspace of l^∞, so that c_0 is complete by 1.5-2 and 1.4-7.

Proof:
Let $(\mathbf{x}_n) \subset c_0$ such that $\mathbf{x}_n \to \mathbf{x}$ for some $\mathbf{x} \in l^\infty$.

Then $\mathbf{x}_1 = (x_{11}, x_{12}, ..., x_{1n}, ...) \to 0.$
$\mathbf{x}_2 = (x_{21}, x_{22}, ..., x_{2n}, ...) \to 0.$
$\cdots$
$\mathbf{x}_m = (x_{m1}, x_{m2}, ..., x_{mn}, ...) \to 0.$
$\cdots$
$\mathbf{x} = (x_1, x_2, ... , x_n, ...)$

Let $\varepsilon > 0$. $\mathbf{x}_n \to \mathbf{x} \Rightarrow \exists\, N \in \mathbb{N} \ni \forall\, m \geq N, \sup_{k \in N}|x_{mk} - x_k| \leq \varepsilon/2.$

Fix $m \geq N$. Then $\exists\, N_{(m)} \in \mathbb{N} \ni \forall\, k \geq N_{(m)}\ |x_{mk} - 0| < \varepsilon/2.$
So then $|x_k - 0| \leq |x_k - x_{mk}| + |x_{mk}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\therefore$ $x_k \to 0$, hence $\mathbf{x} \in c_0$.

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3. In l^∞, let Y be the subset of all sequences with only finitely many nonzero terms. Show that Y is a subspace of l^∞ but not a closed subspace.

Proof:
Clearly $\Theta = (0, 0, ...) \in Y$.
Let $\mathbf{x}, \mathbf{y} \in Y$ and let $\alpha \in K$. Then $\mathbf{x} = (x_1, x_2, ..., x_n, 0, 0, ...)$ and $\mathbf{y} = (y_1, y_2, ..., y_m, 0, 0, ...)$ where n, m are the largest indices for which the terms of $\mathbf{x}$ and $\mathbf{y}$ are non-zero, repectively. Note that some of the x_i and y_i may be 0 for $1 \leq i \leq r = \max\{n, m\}$.
Then $\alpha\mathbf{x} + \mathbf{y} = (\alpha x_1 + y_1, \alpha x_2 + y_2, ..., \alpha x_r + y_r, 0, 0, ...) \in Y$.

To show Y is not closed, let $(\mathbf{x}_1) = (1, 0, 0, ...)$
$(\mathbf{x}_2) = (1, 1/2, 0, 0, ...)$
$\vdots$
$(\mathbf{x}_n) = (1, 1/2, 1/3, ..., 1/n, 0, 0, ...)$

Then $\mathbf{x}_n \to \mathbf{x}$ where $\mathbf{x} = (1/n)_{n\in\mathbb{N}} \notin Y$. $\therefore$ l^∞ is not a closed subspace of Y.

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5. (a) Show that $x_n \to x$ and $y_n \to y \Rightarrow x_n + y_n \to x + y$.
(b) Show that $\alpha_n \to \alpha$ and $x_n \to x \Rightarrow \alpha_n x_n \to \alpha x$.

Proof:
(a) Suppose $x_n \to x$ and $y_n \to y$. Then $\exists\, N \in \mathbb{N} \ni \forall\, n \geq N, ||x_n - x||, ||y_n - y|| < \varepsilon/2$.
So then $||x_n + y_n - (x + y)|| = ||x_n - x + y_n - y|| \leq ||x_n - x|| + ||y_n - y|| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
(b) Suppose $\alpha_n \to \alpha$ and $x_n \to x$. Note that convergent sequences are bounded (Theorem 1.4-2), hence $\exists\, M \in \mathbb{R} \ni \forall\, n \in \mathbb{N}, ||\, x_n - \Theta\, || = ||\, x_n\, || \leq M$.
$\exists\, N \in \mathbb{N} \ni \forall\, n \geq N, |\alpha_n - \alpha| < \dfrac{\varepsilon}{2M}, |x_n - x| < \dfrac{\varepsilon}{2}$. So then

$$|\alpha_n x_n - \alpha x| \leq |\alpha_n x_n - \alpha x_n| + |\alpha x_n - \alpha x| = |\alpha_n - \alpha||x_n| + |\alpha||x_n - x| < \frac{\varepsilon}{2M} \bullet M + \frac{\varepsilon}{2} = \varepsilon.$$

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6. Show that the closure $\overline{Y}$ of a subspace Y of a normed space X is again a vector subspace.

Proof:
$\Theta \in Y \subset \overline{Y}$ so then $\overline{Y} \neq \varnothing$.
Let $x, y \in Y'$ (the set of all limit points of Y). And let $\alpha \in K$.
Then $\forall\, n \in \mathbb{N}, \exists\, x_n \in Y \ni x_n \neq x$ and $\| x - x_n \| < \dfrac{1}{2\alpha n}$ and
$\exists\, y_n \in Y \ni y_n \neq y$ and $\| y - y_n \| < \dfrac{1}{2n}$.
Thus $\| \alpha x + y - (\alpha x_n + y_n) \| \leq |\alpha| \| x - x_n \| + \| y - y_n \| < |\alpha| \bullet [\varepsilon/(2\alpha n)] + \varepsilon/(2n) < \varepsilon/n$.
So then $\exists\, z_n = \alpha x_n + y_n \in B(\alpha x + y, 1/n)^* \cap Y$.
$\therefore\ \alpha x + y \in Y'$, hence $\overline{Y}$ is a subspace of X.

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9. Show that in a Banach space, an absolutely convergent series is convergent.

Proof:
Let $(X, \|\bullet\|)$ be a Banach space. Let $\varepsilon > 0$.
Let $(x_n) \subset X \ni \sum_{n=1}^{\infty} \|x_n\|$ converges. Then $\exists\, N \in \mathbb{N} \ni n \geq m \geq N \Rightarrow \sum_{i=m+1}^{n} \|x_i\| < \varepsilon$.
Let $s_k = \sum_{n=1}^{k} x_n = x_1 + x_2 + \cdots + x_n$. Then $\| s_n - s_m \| = \sum_{i=m+1}^{n} \|x_i\| < \varepsilon$.
Thus, (s_k) is Cauchy. Since X is Banach, $s_k \to x \in X$,
Then $\lim_{k \to \infty} s_k = \sum_{n=1}^{\infty} x_n = x$. $\therefore \sum_{n=1}^{\infty} x_n$ converges.

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10. Show that if a normed space has a Schauder basis, it is separable.

Proof:
Let $(X, \|\bullet\|)$ be a normed space with a Schauder basis $\{e_1, e_2, \ldots\}$.
Let $\varepsilon > 0$. Let $x \in X$. Then $\exists\, (\alpha_n) \in K$ and $\exists\, N \in \mathbb{N} \ni \| x - \Sigma_{n=1}^{N} \alpha_n e_n \| < \dfrac{\varepsilon}{2}$
By the density property of $\mathbb{C}$, for each fixed n,
$\exists\, r_n = a_n + ib_n \in \{a + ib \mid a, b \in \mathbb{Q}\} \ni |r_n - \alpha_n| < \dfrac{\varepsilon}{2N\|e_n\|}$.

$$
\begin{aligned}
\text{So then } \| x - \Sigma_{n=1}^{N} r_n e_n \| &= \| x - \Sigma_{n=1}^{N} r_n e_n + \Sigma_{n=1}^{N} \alpha_n e_n - \Sigma_{n=1}^{N} \alpha_n e_n \| \\
&\leq \| x - \Sigma_{n=1}^{N} \alpha_n e_n \| + \| \Sigma_{n=1}^{N} (\alpha_n - r_n) e_n \| \\
&= \| x - \Sigma_{n=1}^{N} \alpha_n e_n \| + \Sigma_{n=1}^{N} |(\alpha_n - r_n)|\, \| e_n \| \\
&< \| x - \Sigma_{n=1}^{N} \alpha_n e_n \| + \Sigma_{n=1}^{N} \frac{\varepsilon}{2N\|e_n\|} \| e_n \| \\
&= \| x - \Sigma_{n=1}^{N} \alpha_n e_n \| + \Sigma_{n=1}^{N} \frac{\varepsilon}{2N} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

$\therefore\ x$ is a limit point of $\{\Sigma_{n=1}^{\infty} (a_n + ib_n) e_n \mid a_n, b_n \in \mathbb{Q}, n \in \mathbb{N}\}$, a countable set dense in X.

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11. Show that (e_n), where $e_n = (\delta_{nj})$, is a Schauder basis for l^p, where $1 \leq p < +\infty$.

Proof:
Note that $\delta_{1j} = (1, 0, 0, \ldots)$, $\delta_{2j} = (0, 1, 0, 0, \ldots)$, …, $\delta_{nj} = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$, … .
Let $\mathbf{x} \in l^p$. Then $\mathbf{x} = (x_n)$ where $\Sigma|x_n|^p < +\infty$.
So then $\mathbf{x} = (\alpha_1, \alpha_2, \ldots) = \Sigma\alpha_n(\delta_{nj}) = \Sigma\, \alpha_n e_n$.
$\therefore$ (δ_{nj}) is a Schauder basis for l^p.
