Section 2.4, page 76 # 1,2,8

1. Give examples of subspaces of l^∞ and l^2 which are not closed.
Examples:
Let $Y = \{(x_n)_{n\in\mathbb{N}} \subset l^\infty \mid \exists\, N \in \mathbb{N} \ni x_n = 0\ \forall\, n \geq N\}$.
Let $W = \{(x_n)_{n\in\mathbb{N}} \subset l^2 \mid \exists\, N \in \mathbb{N} \ni x_n = 0\ \forall\, n \geq N\}$.

Proof:
To show Y is not closed, let $(\mathbf{x}_1) = (1, 0, 0, \ldots)$
$(\mathbf{x}_2) = (1, 1/2, 0, 0, \ldots)$
$\vdots$
$(\mathbf{x}_n) = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$
Then $\mathbf{x}_n \to \mathbf{x}$ where $\mathbf{x} = (1/n)_{n\in\mathbb{N}} \notin Y$. $\therefore$ l^∞ is not a closed subspace of Y.
To show W is not closed, consider the same sequence as above.
Note that for each $n \in \mathbb{N}$, $\mathbf{x}_n \in W$ as $\Sigma\mathbf{x}_n < \Sigma(1/n^2)$, a convergent series.

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2. (a) What is the largest possible c in (1) if $X = \mathbb{R}^2$ and $x_1 = (1, 0)$, $x_2 = (0, 1)$?
(b) If $X = \mathbb{R}^3$ and $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$?

(a) Largest possible c: $1/\sqrt{2}$

Proof:
Let $(X, \|\bullet\|) = (\mathbb{R}^2, \|x\| = (|\xi_1|^2 + |\xi_2|^2)^{1/2})$.
By 2.4-1 Lemma, $\exists\, c > 0 \ni \forall\, \alpha_1, \alpha_2 \in K$, $\|\alpha_1 x_1 + \alpha_2 x_2\| \geq c(|\alpha_1| + |\alpha_2|)$. So then
$\|\alpha_1(1, 0) + \alpha_2(0, 1)\| = \|(\alpha_1, 0) + (0, \alpha_2)\| = \|(\alpha_1, \alpha_2)\| = \sqrt{|\alpha_1|^2 + |\alpha_2|^2} \geq c(|\alpha_1| + |\alpha_2|)$.
Thus, $\dfrac{\sqrt{|\alpha_1|^2 + |\alpha_2|^2}}{|\alpha_1| + |\alpha_2|} \geq c$. By exercise #3, Ch 1.2, we know $(|\alpha_1| + |\alpha_2|)^2 \leq 2(|\alpha_1|^2 + |\alpha_2|^2)$.
So then we have $\dfrac{\sqrt{|\alpha_1|^2 + |\alpha_2|^2}}{|\alpha_1| + |\alpha_2|} \geq \dfrac{1}{\sqrt{2}}$. Thus, $c \leq \dfrac{1}{\sqrt{2}}$.

(b) Largest possible c: $1/\sqrt{3}$

Proof:
Let $(X, \|\bullet\|) = (\mathbb{R}^3, \|x\| = (|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2)^{1/2})$
By 2.4-1 Lemma, $\exists\, c > 0 \ni \forall\, \alpha_1, \alpha_2, \alpha_3 \in K$, $\|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\| \geq c(|\alpha_1| + |\alpha_2| + |\alpha_3|)$.
So then $\|(\alpha_1, \alpha_2, \alpha_3)\| = \sqrt{|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2} \geq c(|\alpha_1| + |\alpha_2| + |\alpha_3|)$.
Thus, $\dfrac{\sqrt{|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2}}{|\alpha_1| + |\alpha_2| + |\alpha_3|} \geq c$.
By exercise #3, Ch 1.2, we know $(|\xi_1| + |\xi_2| + |\xi_3|)^2 \leq 3(|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2)$.
So then we have $\dfrac{\sqrt{|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2}}{|\alpha_1| + |\alpha_2| + |\alpha_3|} \geq \dfrac{1}{\sqrt{3}}$. Thus, $c \leq \dfrac{1}{\sqrt{3}}$.

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8. Show that the norms $\|\bullet\|_1$, $\|x\|_1 = |\xi_1| + |\xi_2| + \cdots + |\xi_n|$ and $\|\bullet\|_2$, $\|x\|_2 = (|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2)^{1/2}$ in Prob. 8, Sec. 2.2, satisfy $\frac{1}{\sqrt{n}}\|x\|_1 \le \|x\|_2 \le \|x\|_1$.

Proof:

By exercise #3, Ch 1.2, we know $(|\xi_1| + |\xi_2| + \cdots + |\xi_n|)^2 \le n(|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2)$.

So then $\frac{1}{\sqrt{n}} \le \frac{\sqrt{|\xi_1|^2 + |\xi_2|^2 + \cdots |\xi_n|^2}}{|\xi_1| + |\xi_2| + \cdots |\xi_n|} = \frac{\|x\|_2}{\|x\|_1}$, or equivalently, $\frac{1}{\sqrt{n}}\|x\|_1 \le \|x\|_2$.

Since $\|x\|_2^2 = |\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2 \le (|\xi_1| + |\xi_2| + \cdots + |\xi_n|)^2, = \|x\|_1^2$, then by square rooting both sides, we have $\|x\|_2 \le \|x\|_1$.

$\therefore \frac{1}{\sqrt{n}}\|x\|_1 \le \|x\|_2 \le \|x\|_1$.