Section 2.5, page 76 # 1, 2 ,3, 4

1. Show that $\mathbb{R}^n$ and $\mathbb{C}^n$ are not compact.

Proof:
Let $x_n = (n, 0, 0, \ldots)_{n\in\mathbb{N}}$. Since $\forall\ M \in \mathbb{R}, \exists\ n \in \mathbb{N} \ni ||x_n|| > M$, then $\mathbb{R}^n$ and $\mathbb{C}^n$ are not bounded. Hence, by theorem 2.6-3 (*In a finite dimensional normed space X, any subset $M \subset X$ is compact $\Leftrightarrow$ M is closed and bounded.*), they are also not compact.

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2. Show that a discrete metric space X (cf. 1.1-8) consisting of infinitely many points is not compact.

Proof:
Let $M \subset X$ such that M is countable.
Then $M = \{x_1, x_2, \ldots\}_{n\in\mathbb{N}}$ and $\forall\ i, j \in \mathbb{N} \ni i \neq j$, $||\ x_i - x_j\ || = 1$.
$\therefore$ (x_n) does not have any convergent subsequence, hence X is not compact.

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3. Give examples of compact and noncompact curves in the plane $\mathbb{R}^2$.
Examples:
Compact: $A = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 = 1\}$
Noncompact: $B = \{(x, y) \in \mathbb{R}^2 |\ y = x^2\}$

Proof:
$\forall\ (x_1, y_1), (x_2, y_2) \in A$, $||(x_1, y_1) - (x_2, y_2)|| \leq 2$, hence A is bounded.
Let $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset A'$, then $\mathbf{x}_n \to \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^2$ where $\mathbf{x}_n = (x_n, y_n)$ and $\mathbf{x} = (x, y)$.
And $\exists\ N \in \mathbb{N} \ni \forall\ n \geq N$, $||\mathbf{x}_n - \mathbf{x}|| = ||(x_n, y_n) - (x, y)|| < \varepsilon$.
Thus, $||\mathbf{x}|| = ||\mathbf{x}_n - \mathbf{x}_n + \mathbf{x}|| \leq ||\mathbf{x}_n|| + ||\mathbf{x}_n - \mathbf{x}|| = 1 + ||\mathbf{x}_n - \mathbf{x}|| < 1 + \varepsilon$.
Since $||\mathbf{x}_n - \mathbf{x}|| \to 0$, then $||\mathbf{x}|| = 1$. $\therefore$ $\mathbf{x} \in A$, hence A is closed.
And this gives us that A is compact.

Let $M > 0$. Then $(M, M^2) \in B$ and $||(M, M^2)|| = \sqrt{M^2 + M^4} > M$, hence B is unbounded.
So then B is not compact.

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4. Show that for an infinite subset M in the space s (cf. 2.2-8) to be compact, it is necessary that there are numbers $\gamma_1, \gamma_2, \ldots$, such that for all $x = (\xi_k(x)) \in M$ we have $|\xi_k(x)| \le \gamma_k$. (It can be shown that the condition is also sufficient for the compactness of M.)

Proof:

Define for each $i \in \mathbb{N}$, $\mathbf{x}_i = ((\xi_n^{(i)})_{n\in\mathbb{N}})$. Let $k \in \mathbb{N}$.

Suppose $\exists\ (\mathbf{x}_n)_{n\in\mathbb{N}} \in M$ where the kth term, $(\xi_k^{(i)})$, of each $\mathbf{x}_i$ forms an unbounded sequence $(\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}, \ldots)$.

By definiton of compact, $(\mathbf{x}_n)$ has a convergent subsequence whose limit is an element of M.

So then $\mathbf{x}_n \to \mathbf{x}$ where $\mathbf{x} = (\xi_k) \in M$.

Thus, $\exists\ N_1 \in \mathbb{N} \ni \forall\ n \ge N_1$, $d(\mathbf{x}_n, \mathbf{x}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\xi_k^{(n)} - \xi_k|}{1 + |\xi_k^{(n)} - \xi_k|} < 1/2^{k+1}$.

And since is $(\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}, \ldots)$ unbounded, $\exists\ N_2 \in \mathbb{N} \ni \forall\ n \ge N_2$, $|\xi_k^{(n)}| > 1 + |\xi_k|$.

Let $n \ge \max\{N_1, N_2\}$. Then $\frac{1}{2^k} \frac{|\xi_k^{(n)} - \xi_k|}{1 + |\xi_k^{(n)} - \xi_k|} < \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\xi_k^{(n)} - \xi_k|}{1 + |\xi_k^{(n)} - \xi_k|} < \frac{1}{2^{k+1}}$.

So then $\frac{|\xi_k^{(n)} - \xi_k|}{1 + |\xi_k^{(n)} - \xi_k|} < \frac{1}{2} \Rightarrow 2|\xi_k^{(n)} - \xi_k| < 1 + |\xi_k^{(n)} - \xi_k| \Rightarrow |\xi_k^{(n)} - \xi_k| < 1$.

But $1 < ||\xi_k^{(n)}| - |\xi_k|| \le |\xi_k^{(n)} - \xi_k|$ a contradiction.

$\therefore\ \forall\ k \in \mathbb{N}$, $(\xi_k^{(i)})_{n\in\mathbb{N}}$ cannot form an unbounded sequence.

$\therefore$ There are numbers $\gamma_1, \gamma_2, \ldots$, such that for all $x = (\xi_k)_{k\in\mathbb{N}} \in M$ and $\forall\ k \in \mathbb{N}$, we have $|\xi_k| \le \gamma_k$.