Section 2.6, page 90 # 1, 2, 3, 7, 8, 12, 13, 14, 15

1. Show that the operators in 2.6-2 (I_x: $X \to X$ defined by $I_X x = x \ \forall\ x \in X$),
2.6-3 ($0: X \to Y$ defined by $0x = 0 \ \forall\ x \in X$), and
2.6-4 (X = {all polynomials on $[a, b]$}, $Tx(t) = x'(t)$ for every $x \in X$) are linear.

Proof:
Let $\alpha, \beta \in K$, and $x, y \in X$. Then $I_X(\alpha x + \beta y) = \alpha x + \beta y = \alpha I_X x + \beta I_X y$.
$0(\alpha x + \beta y) = 0 = \alpha 0(x) + \beta 0(y)$.
$T(\alpha x + \beta y) = \alpha x' + \beta y' = \alpha Tx + \beta Ty$.
$\therefore$ Each operator is linear.

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2. Show that the operators $T_1, \ldots, T_4$ from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by
$(\xi_1, \xi_2) \mapsto (\xi_1, 0)$
$(\xi_1, \xi_2) \mapsto (0, \xi_2)$
$(\xi_1, \xi_2) \mapsto (\xi_2, \xi_1)$
$(\xi_1, \xi_2) \mapsto (\gamma\xi_1, \gamma\xi_2)$
respectively, are linear, and interpret these operators geometrically.

Proof:
Let $\alpha, \beta \in K$, and $(\xi_1, \xi_2), (\eta_1, \eta_2) \in \mathbb{R}^2$. Then
$T_1(\alpha(\xi_1, \xi_2) + \beta(\eta_1, \eta_2)) = T_1((\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2))$
$= (\alpha\xi_1 + \beta\eta_1, 0)$
$= \alpha(\xi_1, 0) + \beta(\eta_1, 0) = \alpha T_1(\xi_1, \xi_2) + \beta T_1(\eta_1, \eta_2)$
Hence, T_1 is linear. T_1 sends elements from $\mathbb{R}^2$ to the horizontal axis.

$T_2(\alpha(\xi_1, \xi_2) + \beta(\eta_1, \eta_2)) = T_2((\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2))$
$= (0, \alpha\xi_2 + \beta\eta_2)$
$= \alpha(0, \xi_2) + \beta(0, \eta_2) = \alpha T_2(\xi_1, \xi_2) + \beta T_2(\eta_1, \eta_2)$
Hence, T_2 is linear. T_2 sends elements from $\mathbb{R}^2$ to the vertical axis.

$T_3(\alpha(\xi_1, \xi_2) + \beta(\eta_1, \eta_2)) = T_3((\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2))$
$= (\alpha\xi_2 + \beta\eta_2, \alpha\xi_1 + \beta\eta_1)$
$= \alpha(\xi_2, \xi_1) + \beta(\eta_2, \eta_1) = \alpha T_3(\xi_1, \xi_2) + \beta T_3(\eta_1, \eta_2)$
Hence, T_3 is linear. T_3 mirrors elements from $\mathbb{R}^2$ across the diagonal line through the origin.

$T_4(\alpha(\xi_1, \xi_2) + \beta(\eta_1, \eta_2)) = T_4((\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2))$
$= (\alpha\gamma\xi_1 + \beta\gamma\eta_1, \alpha\gamma\xi_2 + \beta\gamma\eta_2)$
$= \alpha(\gamma\xi_1, \gamma\xi_2) + \beta(\gamma\eta_1, \gamma\eta_2) = \alpha T_4(\xi_1, \xi_2) + \beta T_4(\eta_1, \eta_2)$
Hence, T_4 is linear. T_4 sends elements of $\mathbb{R}_2$ distance of γ along the diagonal through each point.

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3. What are the domain, range and null space of T_1, T_2, T_3 in Prob. 2?
$\mathscr{D}(T_1) = \mathbb{R}^2$, $\mathscr{R}(T_1) = \{(x, 0) | x \in \mathbb{R}\}$, $\mathscr{N}(T_1) = \{(0, x) | x \in \mathbb{R}\}$,
$\mathscr{D}(T_2) = \mathbb{R}^2$, $\mathscr{R}(T_2) = \{(0, y) | y \in \mathbb{R}\}$, $\mathscr{N}(T_2) = \{(x, 0) | x \in \mathbb{R}\}$,
$\mathscr{D}(T_3) = \mathbb{R}^2$, $\mathscr{R}(T_3) = \mathbb{R}^2$, $\mathscr{N}(T_3) = \{(0, 0)\}$.

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7. Let X be any vector space and S: X → X and T: X → X any operators. S and T are said to commute if ST = TS, that is, (ST)x = (TS)x for all x ∈ X. Do T_1 and T_3 in Prob. 2 commute?

Yes.

Proof:
$\forall\ (x, y) \in \mathbb{R}^2$, $T_1 T_3 ((x, y)) = T_1(y, x) = (0, x)$; and $T_3 T_1 ((x, y)) = T_1(x, 0) = (0, x)$.

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8. Write the operators in Prob. 2 using 2 × 2 matrices.

$$T_1((x, y)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

$$T_2((x, y)) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

$$T_3((x, y)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

$$T_4((x, y)) = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \gamma x \\ \gamma y \end{pmatrix}.$$

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12. Does the inverse of T in 2.6-4(X = {all polynomials on $[a, b]$}, $Tx(t) = x'(t)$ for every $x \in X$) exist?

No.

Proof:
Let $\alpha, \beta \in K \ni \alpha \neq \beta$. Then $T(\alpha) = T(\beta) = 0$. Thus, $\mathscr{N}(T) \neq \{0\}$, hence T^{-1} does not exist.

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13. Let T: $\mathscr{D}(T) \to Y$ be a linear operator whose inverse exists. If $\{x_1, ..., x_n\}$ is a linearly independent set in $\mathscr{D}(T)$, show that the set $S = \{Tx_1, ..., Tx_n\}$ is linearly independent.

Proof:
Assume S is linearly dependent. Then
$\Theta_Y = \alpha_1 T(x_1) + \alpha_2 T(x_2) + \cdots + \alpha_n T(x_n)$ for some $\alpha_1, \alpha_2, ..., \alpha_n \in K$, and not all $\alpha_i = 0$.
So then $T^{-1}(\Theta_Y) = \Theta_X = T^{-1}(\alpha_1 T(x_1) + \alpha_2 T(x_2) + \cdots + \alpha_n T(x_n)) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$.
This contradicts that $\{x_1, ..., x_n\}$ is linearly independent.
$\therefore$ S is linearly independent also.

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14. Let T: $X \to Y$ be a linear operator and dim X = dim $Y = n < \infty$.
Show that $\mathscr{R}(\mathrm{T}) = Y \Leftrightarrow \mathrm{T}^{-1}$ exists.

Proof:
$\Rightarrow$: Let $B = \{x_1, ..., x_n\}$ be a basis for X.
Suppose T^{-1} does not exist. Then $\mathscr{N}(\mathrm{T}) \neq \{\Theta_X\}$. Thus, $\exists\, x \neq \Theta_X \in X \ni \mathrm{T}(x) = \Theta_Y$.
And $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ for some $\alpha_1, \alpha_2, ..., \alpha_n \in K$ where not all $\alpha_i = 0$.
So then $\mathrm{T}(x) = \Theta_Y = \alpha_1 \mathrm{T}(x_1) + \alpha_2 \mathrm{T}(x_2) + \cdots + \alpha_n \mathrm{T}(x_n)$.
Thus $\{\mathrm{T}x_1, \mathrm{T}x_2, ..., \mathrm{T}x_n\}$ is a linearly dependent set.
Consider $y \in \mathscr{R}(\mathrm{T})$.
Then $\exists\, x \in X \ni T(x) = y$ and $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ for some $\alpha_1, \alpha_2, ..., \alpha_n \in K$.
$\mathrm{T}(x) = \alpha_1 \mathrm{T}(x_1) + \alpha_2 \mathrm{T}(x_2) + \cdots + \alpha_n \mathrm{T}(x_n)$, hence span $\{\mathrm{T}x_1, \mathrm{T}x_2, ..., \mathrm{T}x_n\} = \mathscr{R}(\mathrm{T})$.
Since $\{\mathrm{T}x_1, \mathrm{T}x_2, ..., \mathrm{T}x_n\}$ spans $\mathscr{R}(\mathrm{T})$ but is linearly independent, then dim $\mathscr{R}(\mathrm{T}) \leq n - 1$.
This gives us that $\mathscr{R}(\mathrm{T}) \neq Y$.
$\Leftarrow$: By contrapositive, suppose $\mathscr{R}(\mathrm{T}) \neq \mathrm{Y}$. Then $\exists\, y \in Y \ni \forall\, x \in X$, $\mathrm{T}(x) \neq y$.
$\therefore$ T^{-1} does not exist.

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15. Consider the vector space X of all real-valued functions which are defined on $\mathbb{R}$ and have derivatives of all orders everywhere on $\mathbb{R}$. Define T: $X \to X$ by $y(t) = \mathrm{T}x(t) = x'(t)$. Show that $\mathscr{R}(\mathrm{T})$ is all of X but T^{-1} does not exist.
Compare with Prob. 14 and comment.

Proof:
To show T is onto, let $y \in X$.
Then $y = f(t)$ for some $f \in \{f : \mathbb{R} \to \mathbb{R} \mid f^{(n)}$ exists for all $n \in \mathbb{N}\}$.
And $\mathrm{T}(\int f(t)dt) = y$. $\therefore$ T is onto.
To show T is not 1-1, we note that $\mathrm{T}(\alpha_0) = 0$ and $\mathrm{T}(\alpha_1) = 0$ where $\alpha_0 \neq \alpha_1$.
In comparison with #14, we see that this exemplifies a case in which a linear operator is onto, but not 1-1. In #14, we used the fact that both the domain and range were of the same finite dimension. In #15, we have a linear operator from a domain to a range both of infinite dimension.
