Section 2.7 p. 101 # 1, 2, 3, 5, 6, 7, 8, 9

**1.** Prove (7) (a) ($\|T_1 T_2\| \le \|T_1\| \|T_2\|$, (b) $\|T^n\| \le \|T\|^n$ valid for bounded linear operators $T_2 : X \to Y$, $T_1 : Y \to Z$ and $T: X \to X$, where $X$, $Y$, $Z$ are normed spaces.).

**Proof**:

(a) $\|T_1 T_2\| = \sup\limits_{\substack{x\in X\\ x\neq 0}} \dfrac{\|T_1T_2x\|}{\|x\|} \le \sup\limits_{\substack{x\in X\\ x\neq 0}} \dfrac{\|T_1\|\|T_2x\|}{\|x\|} = \|T_1\|\sup\limits_{\substack{x\in X\\ x\neq 0}} \dfrac{\|T_2x\|}{\|x\|} = \sup\limits_{\substack{y\in Y\\ y\neq 0}} \dfrac{\|T_1y\|}{\|y\|} \bullet \sup\limits_{\substack{x\in X\\ x\neq 0}} \dfrac{\|T_2x\|}{\|x\|} = \|T_1\| \|T_2\|$

(b) By (a), $\|T T\| \le \|T\|^2$. Let $n > 1$ and assume $\|T^n\| \le \|T\|^n$.
Then $\|T^{n+1}\| = \|T T^n\| \le \|T\| \|T\|^n = \|T\|^{n+1}$.

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**2.** Let $X$ and $Y$ be normed spaces. Show that a linear operator
$T: X \to Y$ is bounded $\Leftrightarrow$ T maps bounded sets in $X$ into bounded sets in $Y$.

**Proof**:

$\Rightarrow$: Assume T is bounded. Then $\exists\, c \in \mathbb{R} \ni \|Tx\| \le c\|x\| \;\forall\, x \in X$.
Let $M \subset X$ be bounded. Let $x_0 \in M$. Then $\exists\, r \in \mathbb{R} \ni \forall\, x \in M$, $\|x - x_0\| < r$.
So then $\|Tx - Tx_0\| = \|T(x - x_0)\| \le c\|x - x_0\| < cr$. $\therefore$ $T(M)$ is bounded.

$\Leftarrow$: Assume T maps bounded sets in $X$ into bounded sets in $Y$.
Let $M = \{x \in X \mid \|x\| = 1\}$. Then $M$ is bounded, hence $T(M)$ is bounded.
$\therefore$ $\exists\, r \in \mathbb{R}^+ \ni \|Tx\| < r \;\forall\, x \in M$.

Note that by 2.7-2 Lemma, $\|T\| = \sup\limits_{\substack{x\in D(T)\\ x\neq 0}} \dfrac{\|Tx\|}{\|x\|} = \sup\limits_{\substack{x\in D(T)\\ \|y\|=1}} \|Ty\|$.

So then $\|Tx\| < r \;\forall\, x \in X \ni \|x\| = 1$

$\Rightarrow \sup\limits_{\substack{x\in D(T)\\ x\neq 0}} \dfrac{\|Tx\|}{\|x\|} < r$

$\Rightarrow \sup\limits_{\substack{x\in D(T)\\ x\neq 0}} \|Tx\| < r\|x\|$

$\Rightarrow \|Tx\| \le r\|x\| \;\forall\, x \in X$.

$\therefore$ T is bounded.

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**3.** If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathscr{D}(T) \ni \|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|$.

**Proof**:

Let $x \in \mathscr{D}(T) \ni \|x\| < 1$. Then $T \neq 0$ and T is bounded, we have
$\|Tx\| \le \|T\| \|x\| < \|T\|$.

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**5.** Show that the operator T: $l^\infty \to l^\infty$ defined by $y = (\eta_j) = Tx$, $\eta_j = \xi_j/j$, $x = (\xi_j)$, is linear and bounded.

**Proof**:
Let $\alpha, \beta \in K$, and $x = (\xi_j)$, $y = (\eta_j) \in l^\infty$. Then
$T(\alpha x + \beta y) = T(\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2, ...) = (\alpha\xi_1/1 + \beta\eta_1/1, \alpha\xi_2/2 + \beta\eta_2/2, ...)$
$= \alpha(\xi_1, \xi_2/2, ...) + \beta(\eta_1, \eta_2/2, ...) = \alpha Tx + \beta Ty$. So T is linear.
To show T is bounded, let $x \in l^\infty$ and note that $\| x \| = \sup_{j\in N}\{|\xi_j|\}$ and $\| Tx \| = \sup_{j\in N}\{|\xi_j / j|\}$.
Since $x$ is a bounded sequence, then $\exists\, r \in \mathbb{R} \ni \forall\, j \in \mathbb{N}$, $|\xi_j| < r$.
And since for each $j$, $|\xi_j/j| \le \xi_j$, then $\sup_{j\in N}\{|\xi_j / j|\} \le \sup_{j\in N}\{|\xi_j|\}$.
$\therefore$ $\| Tx \| \le \| x \|$, hence T is bounded.

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**6.** Show that the range $\mathscr{R}(T)$ of a bounded linear operator T: $X \to Y$ need not be closed in $Y$. *Hint*. Use T in Prob. 5.
**Proof**:
Let T: $l^\infty \to l^\infty$ be defined by $T\mathbf{x} = T(\xi_j) = \xi_j/j$. From exercise #5 above, we know this operator is bounded.
Let $M = \{\mathbf{x} \in l^\infty \mid \mathbf{x}$ has finitely many nonzero terms$\}$. Then $\forall\, \mathbf{y} \in T(M)$, $\mathbf{y}$ has finitely many nonzero terms.
Consider $T|_M : M \to l^\infty$.
Let $\mathbf{x}_1 = (1, 0, 0, ...)$, $\mathbf{x}_2 = (1, 1, 0, 0, ...)$, ..., $\mathbf{x}_n = (1, 1, 1, ..., 1, 0, 0, ...)$, ... . Then
$T(\mathbf{x}_1) = (1, 0, 0, ...)$, $T(\mathbf{x}_2) = (1, 1/2, 0, 0, ...)$, ..., $T(\mathbf{x}_n) = (1, 1/2, 1/3, ..., 1/n, 0, 0, ...)$, ... .
So then $T(\mathbf{x}_n) \to (1, 1/2, 1/3, ..., 1/n, 1/(n+1), ...) \notin T(M) = \mathscr{R}(T|_M)$.
Hence $\mathscr{R}(T|_M)$ is not closed in $l^\infty$.

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**7.** Let T be a bounded linear operator from a normed space $X$ onto a normed space $Y$. If there is a positive $b$ such that $\|Tx\| \ge b\|x\|$ for all $x \in X$, show that then $T^{-1}: Y \to X$ exists and is bounded.

**Proof**:
By assumption, $\exists\, b, c \in \mathbb{R}^+ \ni b\|x\| \le \|Tx\| \le c\|x\|$.
Let $x \in \mathscr{N}(T)$. Then $Tx = \Theta$. If $x \ne \Theta$, then $\|x\| \ne 0$.
But $0 < b\|x\| \le 0 \le c\|x\|$, clearly a contradiction.
$\therefore$ $N(T) = \{\Theta\}$, hence $T^{-1}$ exists.
Let $y \in Y$, then $\exists\, x \in X \ni Tx = y$. Substituting $y$ for $Tx$ in $b\|x\| \le \|Tx\| \le c\|x\|$ we have $b\|T^{-1}y\| \le \|y\| \le c\|T^{-1}y\|$.
$\therefore$ $\| T^{-1}y \| \le (1/b)\, \| y \|$, hence $T^{-1}$ is bounded.

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**8.** Show that the inverse $T^{-1}: \mathscr{R}(T) \to X$ of a bounded linear operator $T: X \to Y$ need not be bounded. *Hint*. Use T in Prob. 5.

**Proof**:
Let $T: l^\infty \to l^\infty$ be defined by $Tx = T(\xi_j) = \xi_j/j$.
Let $\mathbf{y}_1 = (1, 0, 0, ...)$, $\mathbf{y}_2 = (0, 1/2, 0, ...)$, ..., $\mathbf{y}_n = (0, 0, ..., 1/n, 0, 0, ...)$, ...
Then $T^{-1}(\mathbf{y}_1) = (1, 0, 0, ...)$, $T^{-1}(\mathbf{y}_2) = (0, 2\bullet(1/2), 0, ...)$, ..., $T^{-1}(\mathbf{y}_n) = (0, 0, ..., n\bullet(1/n), 0, ...)$, ... .
Let $r \in \mathbb{R}^+$.
Then $\exists\ n > r \ni n\bullet|\xi_n| = \max\{|j\bullet\xi_j|\}_{j\in\mathbb{N}} = ||T^{-1}\mathbf{y}_n|| > r\,||\mathbf{y}_n|| = r \bullet \max\{|\xi_j|\}_{j\in\mathbb{N}} = r\,|\xi_n|$.
$\therefore$ $T^{-1}$ is not bounded.

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**9.** Let $T: \mathscr{C}[0, 1] \to \mathscr{C}[0, 1]$ be defined by $y = Tx$ where $y(t) = \int_0^t x(\tau)d\tau$.
Find $\mathscr{R}(T)$ and $T^{-1}: \mathscr{R}(T) \to \mathscr{C}[0, 1]$. Is $T^{-1}$ linear and bounded?

$\mathscr{R}(T) = \{(x(t), \int_0^t x(\tau)d\tau) \in \mathscr{C}[0, 1]\times\mathscr{C}[0, 1] \mid x$ is differentiable and $x(0) = 0\}$

$T^{-1}: \mathscr{R}(T) \to \mathscr{C}[0, 1] = \{(h(t), h'(t)) \in \mathscr{C}[0, 1]\times\mathscr{C}[0, 1]\}$

$T^{-1}$ is linear but not bounded.

**Proof**:
Let $x(t) \in \mathscr{C}[0, 1]$. Then $\exists\ y(t) \in \mathscr{C}[0, 1] \ni y'(t) = x(t)$.
$T^{-1}(Tx) = T^{-1}(y(t) - y(0)) = y'(t) = x(t)$.
Let $y(t) \in \mathscr{C}[0, 1]$.
And $T(T^{-1}(y(t))) = T(y'(t)) = \int_0^t y'(\tau)d\tau = y(t) - y(0) = y(t)$.

Let $\alpha, \beta \in K$, and $x, y \in \mathscr{R}(T)$.
Then $T^{-1}(\alpha x + \beta y) = \alpha x' + \beta y' = \alpha T^{-1}x + \beta T^{-1}y$, hence $T^{-1}$ is linear.
To show $T^{-1}$ is not bounded, let $n \in \mathbb{N}$ and notice that

$||T^{-1}(t^n)|| = n||(t^{n-1})|| = \frac{n\|t^n\|}{\|t\|}$. Since $||T^{-1}|| = \sup_{\substack{x\in D(T)\\ x\neq 0}} \frac{\|Tx\|}{\|x\|}$, then $||T^{-1}|| \geq n$.

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