Section 2.10 p. # 4, 8, 9, 10, 11

4. Let X and Y be normed spaces and $T_n: X \to Y$ ($n = 1, 2, \ldots$) bounded linear operators. Show that convergence $T_n \to T$ implies that for every $\varepsilon > 0$ there is an N such that for all $n > N$ and all x in any given closed ball we have $\|T_n x - Tx\| < \varepsilon$.

Proof:
Let $\varepsilon > 0$, $r \in \mathbb{R}^+$, and $x \in B = \{x \in X \mid \|x\| \le r\}$.
$T_n \to T \Rightarrow \exists\, N \in \mathbb{N} \ni n \ge N \Rightarrow \|T_n - T\| < \varepsilon/r$. Let $n \ge N$.
So then if $T_n - T$ is a bounded linear operator, we have
$\|T_n x - Tx\| = \|(T_n - T)x\| \le \|T_n - T\|\,\|x\| < (\varepsilon/r) \bullet r = \varepsilon$, as desired. ($*$)
We only need to verify $T_n - T$ is bounded and linear.
To show $T_n - T$ is linear, we let $\alpha, \beta \in K$ and $x, y \in X$ and note that

$$\begin{aligned}(T_n - T)(\alpha x + \beta y) &= T_n(\alpha x + \beta y) + \lim_{n\to\infty}(T_n(\alpha x + \beta y)) \\ &= T_n(\alpha x + \beta y) + \lim_{n\to\infty}(T_n(\alpha x)) + \lim_{n\to\infty}(T_n(\beta y)) \\ &= \alpha\left(T_n x + \lim_{n\to\infty}(T_n(x))\right) + \beta\left(T_n y + \lim_{n\to\infty}(T_n(y))\right) \\ &= \alpha(T_n - T)x + \beta(T_n - T)y.\end{aligned}$$

To show $T_n - T$ is bounded, we need to show that
$\forall\, x \in X$, $\|(T_n - T)x\| \le c\|x\|$ for some $c \in \mathbb{R}^+$.
Notice that $T_n \to T \Rightarrow (T_n)$ is Cauchy. So $\exists\, N_2 \in \mathbb{N} \ni n, m \ge N_2 \Rightarrow \|T_n - T_m\| < 1$.
Let $n, m \ge N_2$.
Then $\forall\, x \in X$, $\|(T_n - T)x\| = \|T_n x - Tx\| = \|T_n x - \lim_{m\to\infty} T_m x\| = \lim_{m\to\infty} \|T_n x - T_m x\| \le 1 \bullet \|x\|$.
$\therefore$ $T_n - T$ is bounded and the result ($*$) is valid.

8. Show that the dual space of the space c_0 is l^1 (c_0 is the space of all sequences converging to 0).

Proof:
We need to show $\exists\, T: (c_0)' \to l^1$ that is linear, continuous, norm-preserving, 1-1, and onto.
Define $Tf = (f(e_1), f(e_2), \ldots)$ where $(e_k) = ((\delta_{jk}))$ is a Schauder basis for c_0 and $f \in (c_0)'$.

(i) To show T is linear, let $\alpha, \beta \in K$, and $f, g \in c_0$ and note that

$$\begin{aligned} T(\alpha f + \beta g) &= ((\alpha f + \beta g)(e_1), (\alpha f + \beta g)(e_2), \ldots) \\ &= (\alpha f(e_1) + \beta g(e_1), \alpha f(e_2) + \beta g(e_2), \ldots) \\ &= (\alpha f(e_1), \alpha f(e_2), \ldots) + (\beta g(e_1), \beta g(e_2), \ldots) \\ &= \alpha(f(e_1), f(e_2), \ldots) + \beta(g(e_1), g(e_2), \ldots) \\ &= \alpha Tf + \beta Tg \end{aligned}$$

(ii) To show T is continuous, let $x = (\alpha_1 e_1, \alpha_2 e_2, \ldots) \in c_0$, and note that $\lim_{n\to\infty} \alpha_n = 0$.
So then for $s_n = \Sigma_{k=1}^{n} \alpha_k e_k$, we have $s_n \to x$.
Let $f \in (c_0)'$. Then, by continuity of f, we have $f(s_n) \to f(x)$.
Since x was arbitrary, this gives us that Tf is continuous on c_0.
Since f was arbitrary, this gives us that T is continuous on $(c_0)'$.

(iii) To show T is norm-preserving, let $x = (\alpha_1 e_1, \alpha_2 e_2, \ldots) \in c_0$ and $f \in (c_0)'$. Then
$|f(x)| = |\Sigma_{i=1}^{\infty} \alpha_i f(e_i)| \le \Sigma_{i=1}^{\infty} |\alpha_i f(e_i)| \le \sup_{n\in N} |\alpha_n| \Sigma_{i=1}^{\infty} |f(e_i)| = \|x\|_\infty \bullet \Sigma_{i=1}^{\infty} |f(e_i)|$, hence
$\|f\| = \sup_{\substack{x\in c_0 \\ \|x\|=1}} |f(x)| \le 1 \bullet \Sigma_{i=1}^{\infty} |f(e_i)| = \|Tf\|_1$.
And now consider $x^{(n)} = (|f(e_1)|/f(e_1), |f(e_2)|/f(e_2), \ldots, |f(e_n)|/f(e_n), 0, 0, \ldots)$,
unless $f(e_k) = 0$, in which case the kth component is 0.
We know $x^{(n)} \in c_0$ for each $n \in \mathbb{N}$ as each has finitely many nonzero terms.
Note that $\|x^{(n)}\|_\infty = 1$ for each n. Then

$$\begin{aligned} |f(x^{(n)})| &= |(|f(e_1)|/f(e_1))f(e_1) + (|f(e_2)|/f(e_2))f(e_2) + \ldots + (|f(e_n)|/f(e_n))f(e_n)| \\ &= |f(e_1)| + |f(e_2)| + \cdots + |f(e_n)|. \end{aligned}$$

And so $\|Tf\|_1 = \Sigma_{n=1}^{\infty} |f(e_n)| = \lim_{n\to\infty} |f(x^{(n)})| \le \lim_{n\to\infty} (\|f\|\,\|x^{(n)}\|_\infty) = \|f\|$.
Together, the two inequalities give us that $\|Tf\| = \|f\|$.

(iv) To show T is 1-1, let $f \in \mathscr{N}(T)$. Then $Tf = 0$ and $\|Tf\| = \|f\| = 0$.
$\therefore f = 0$, hence T is 1-1.

(v) To show T is onto, let $(\beta_1, \beta_2, \ldots) \in l^1$ and define $g : c_0 \to \mathbb{R}$ by
$g(e_n) = \beta_n$ for each $n \in \mathbb{N}$.
Extend this linearly to $g(x) = \Sigma_{n=1}^{\infty} \alpha_n g(e_n) = \Sigma_{n=1}^{\infty} \alpha_n \beta_n$ where $x = (\alpha_1 e_1, \alpha_2 e_2, \ldots) \in c_0$,
We need to verify that $g \in (c_0)'$. Clearly g is linear.
To show g is bounded, note that $|g(x)| = |\Sigma_{n=1}^{\infty} \alpha_n \beta_n| \le \sup_{n\in N} |\alpha_n| \Sigma_{n=1}^{\infty} |\beta_n| < +\infty$.
$\therefore g \in (c_0)'$ and $Tg = (\beta_1, \beta_2, \ldots)$, hence T is onto.

9. Show that a linear functional f on a vector space X is uniquely determined by its values on a Hamel basis for X. (Cf. Sec. 2.1).

Proof:
Let $\{e_1, e_2, ...\}$ be a Hamel basis for X.
Let $f, g \in X'$ such that $f(e_i) = g(e_i)$ for each i.
Let $x = (\alpha_1 e_1, \alpha_2 e_2, ..., \alpha_n e_n) \in X$, $n \in \mathbb{N}$.
Then $f(x) = \Sigma_{k=1}^{n} \alpha_k f(e_k) = \Sigma_{k=1}^{n} \alpha_k g(e_k) = g(x)$.
$\therefore f = g$.

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10. Let X and $Y \neq \{0\}$ be normed spaces, where $\dim X = \infty$. Show that there is at least one unbounded linear operator T: $X \to Y$. (Use a Hamel basis.)

Proof:
Let $B = \{e_1, e_2, ...\}$ be a Hamel basis for X.
Let $x = (\alpha_1 e_1, \alpha_2 e_2, ..., \alpha_n e_n) \in X$. Define $T_n: X \to \mathbb{R}$ by $T_n(x) = \begin{cases} \alpha_i \|e_i\| \text{ if } i = n \\ 0 \text{ otherwise} \end{cases}$.
Due to uniqueness of representation of each x, T_n is well-defined.
Choose $y \in Y \ni \| y \| = 1$ and define $T: X \to Y$ by $T(x) = (\Sigma_{k=1}^{\infty} nT_k(x))y$.
Since, for each $x \in X$, $(\Sigma_{k=1}^{\infty} nT_k(x))y$ is a sum of finitely many non-zero terms, $T(x) \in Y$.
And note that $\dim X = \infty \Rightarrow \forall\, n \in \mathbb{N}, e_n \in B$.
Let $r \in \mathbb{R}^+$. Then $\exists\, n \in \mathbb{N} \ni n > r$ and $Te_n = n||e_i||y$, hence $|| Te_n || = ||n \bullet ||e_n|| \bullet y|| = n||e_n||$.
To show T is linear,
Let $\lambda, \gamma \in K$ and $x = (\alpha_1 e_1, \alpha_2 e_2, ..., \alpha_n e_n)$, $y = (\beta_1 e_1, \beta_2 e_2, ..., \beta_m e_m) \in X$. Then, for each k,
$T_k(\lambda x + \gamma y) = (\lambda\alpha_k + \gamma\beta_k)||e_k|| = \lambda\alpha_k||e_k|| + \gamma\beta_k||e_k||$. So then
$T(\lambda x + \gamma y) = (\Sigma_{k=1}^{\infty} nT_k(\lambda x + \gamma y))y$
$= (\Sigma_{k=1}^{\infty} n(\lambda\alpha_k||e_k|| + \gamma\beta_k||e_k||))y$
$= \lambda(\Sigma_{k=1}^{\infty} n\alpha_k||e_k||)y + \gamma(\Sigma_{k=1}^{\infty} n\beta_k||e_k||)y$
$= \lambda(\Sigma_{k=1}^{\infty} nT_k(x))y + \gamma(\Sigma_{k=1}^{\infty} nT_k(\gamma y))y = \lambda T(x) + \gamma T(y)$.
$\therefore$ T is an unbounded linear operator from X to Y.

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11. If X is a normed space and $\dim X = \infty$, show that the dual space X' is not identical with the algebraic dual space X^*.

Proof:
Let $B = \{e_1, e_2, ...\}$ be a Hamel basis for X.
Since $\dim X = \infty$, then for each $n \in \mathbb{N}$, $e_n \in B$.
Let $x = (\alpha_1 e_1, \alpha_2 e_2, ..., \alpha_n e_n) \in X$. Define $f_n(x) = \begin{cases} \alpha_i \|e_i\| \text{ if } i = n \\ 0 \text{ otherwise} \end{cases}$.
Define $f \in X^*$ by $f(x) = \Sigma_{k=1}^{\infty} nf_k(x)$. Then $fe_n = n||e_n||$ hence f is unbounded.
Since X' consists only of bounded linear functionals, then no isomorphism exists between X^* and X'.
