Section 2.8 p. # 2, 12, 15

2. Show that the functionals defined on $\mathscr{C}[a, b]$ by

$f_1(x) = \int_a^b x(t)y_0(t)dt$ ($y_0 \in \mathscr{C}[a, b]$) and $f_2(x) = \lambda x(a) + \eta x(b)$ (λ, η fixed)

are linear and bounded.

Proof:
Let $x, y \in \mathscr{C}[a, b]$ and $\alpha, \beta \in K$.
$f_1(\alpha x + \beta y) = \int_a^b (\alpha x + \beta y)(t)y_0(t)dt = \alpha\int_a^b x(t)y_0(t)dt + \beta\int_a^b y(t)y_0(t)dt = \alpha f_1(x) + \beta f_1(y)$.
$f_2(\alpha x + \beta y) = \lambda(\alpha x + \beta y)(a) + \eta(\alpha x + \beta y)(b) = \alpha(\lambda x(a) + \eta x(b)) + \beta(\lambda y(a) + \eta y(b)) = \alpha f_2(x) + \beta f_2(y)$.
Thus, f_1 and f_2 are linear. Let $x \in X$.
$\| f_1 x\| = \max_{a \le t \le b}\left|\int_a^b x(t)dt\right| \le \max_{a \le t \le b}\int_a^b |x(t)|dt \le \max_{a \le t \le b}|x(t)||b - a| = \|x\||b - a|$.
$\| f_2 x\| = |\lambda x(a) + \eta x(b)| \le |\lambda x(a)| + |\eta x(b)| \le \lambda\|x\| + \eta\|x\| = (\lambda + \eta)\|x\|$.
So then f_1 and f_2 are bounded.

---

12. If Y is a subspace of a vector space X and codim $Y = 1$ ($X/Y = \{x + Y | x \in X\}$ and codim Y = dim(X/Y), then every element of X/Y is called a hyperplane parallel to Y. Show that for any linear functional $f \neq 0$ on X, the set $H_1 = \{x \in X \mid f(x) = 1\}$ is a hyperplane parallel to the null space $\mathscr{N}(f)$ of f
(that is, $H_1 = \{x_0 + \mathscr{N}(f) |$ for some $x_0 \in X\}$)

Proof:
Let $f: X \to \mathbb{R}$ be a nonzero linear functional. Let dim $X = n$.
We first will show dim $X/\mathscr{N}(f) = 1$.
We know dim$X/\mathscr{N}(f) =$ dim $X -$ dim$\mathscr{N}(f)$.
Since $f \neq 0$ and $\mathscr{N}(f)$ is a subspace of X, then $X \neq \mathscr{N}(f)$, hence dim $\mathscr{N}(f) \le n - 1$.
Suppose $\mathscr{N}(f) < n - 1$. Let $\{x_1, ..., x_k\}$ be a basis for $\mathscr{N}(f)$.
Then we can extend this to a basis $\{x_1, ..., x_k, x_{k+1}, ..., x_n\}$ for X.
$f(x_{k+1}) = a$ and $f(x_{k+2}) = b$ for some $a, b \in \mathbb{R}^+$.
Thus, by properties of linear operators $f(1/a)x_{k+1}) = 1 = f((1/b)x_{k+2})$.
$\Rightarrow f(1/a)x_{k+1}) - f((1/b)x_{k+2}) = f(1/a)x_{k+1} - (1/b)x_{k+2}) = 0$
$\Rightarrow (1/a)x_{k+1} - (1/b)x_{k+2} \in \mathscr{N}(f)$
$\Rightarrow (1/a)x_{k+1} - (1/b)x_{k+2} = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$ for some $\alpha_1, \alpha_2, ..., \alpha_k \in K$.
$\Rightarrow x_{k+1} = a((1/b)x_{k+2} + \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k)$ where not all coefficients are 0.
$\Rightarrow \{x_1, ..., x_k, x_{k+1}, ..., x_n\}$ is not linearly independent, a contradiction.
Thus, dim$\mathscr{N}(f) = n - 1$, hence dim$X/\mathscr{N}(f) = 1$.
Let $x_0 \in H_1$. Then $f(x_0) \neq 0$, hence $x_0 \neq 0$. Let $x \in X$ and let $y = x - f(x) \bullet x_0$.
Then $f(y) = f(x - f(x) \bullet x_0) = f(x) - f(f(x) \bullet x_0) = f(x) - f(x) \bullet f(x_0) = f(x) - f(x) = 0$.
Thus, $y \in \mathscr{N}(f)$. And $x = f(x)\, x_0 + y \in x_0 + \mathscr{N}(f)$.
Let $x \in x_0 + \mathscr{N}(f)$. Then $x = x_0 + y$ and $f(x) = f(x_0 + y) = f(x_0) + f(y) = 1 + 0 = 1$.
$\therefore$ $x \in H_1$. So then $H_1 = x_0 + \mathscr{N}(f)$.

---

15. Let $f \neq 0$ be a bounded linear functional on a real normed space X. Then for any scalar c we have a hyperplane $H_c = \{x \mid f(x) = c\}$ and H_c determine two half spaces $X_{c1} = \{x \mid f(x) \leq c\}$ and $X_{c2} = \{x \mid f(x) \geq c\}$.
Show that the closed unit ball lies in X_{c1} where $c = \|f\|$, but for no $\varepsilon > 0$, the half space X_{c1} with $c = \|f\| - \varepsilon$ contains that ball.

Proof:

Let $B = \{x \in X \mid \|x\| \leq 1\}$. Let $x \in B$. Let $c = \|f\| = \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in D(f) \\ \|x\|=1}} |f(x)|$.

Note that $f \neq 0 \Rightarrow c > 0$.
Since $f(x) \leq \sup_{\substack{x \in D(f) \\ \|x\|=1}} |f(x)|$, then $f(x) \leq c$, hence $x \in X_{c1}$.

Let $\varepsilon > 0 \ni \varepsilon < \|f\|$. Then by definition of supremum,
$\exists\, x \in X \ni \|x\| = 1$ and $\|f\| - \varepsilon < |f(x)| \leq \|f\|$, hence $f(x) \leq c$, which implies $x \in X_{c1}$.
$\therefore B \not\subset X_{c1}$ where $c = \|f\| - \varepsilon$, $\forall\, \varepsilon > 0$.