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Chapter 2

Definition **2.2-1** A *normed space* is a vector space with a norm defined on it. A *Banach space* is a complete normed space.

Examples Banach spaces

	norm	metric
$\mathbb{R}^n$	$\|\| x \|\| = \left(\sum_{j=1}^n \|\xi_j\|^2\right)^{1/2}$	$d(x,y) = \|\|x-y\|\| = \left(\sum_{j=1}^n \|\xi_j - \eta_j\|^2\right)^{1/2}$
$\mathbb{C}^n$	$\|\| x \|\| = \left(\sum_{j=1}^n \|\xi_j\|^2\right)^{1/2}$	$d(x,y) = \|\|x-y\|\| = \left(\sum_{j=1}^n \|\xi_j - \eta_j\|^2\right)^{1/2}$
l^p	$\|\| x \|\| = \left(\sum_{j=1}^\infty \|\xi_j\|^p\right)^{1/p}$	$d(x,y) = \|\|x-y\|\| = \left(\sum_{j=1}^\infty \|\xi_j - \eta_j\|^p\right)^{1/p}$
l^∞	$\|\| x \|\| = \sup_j \|\xi_j\|$	$d(x,y) = \|\|x-y\|\| = \sup_j \|\xi_j - \eta_j\|$
$\mathscr{C}[a,b]$	$\|\| f \|\| = \max_{a \le x \le b} \|f(x)\|$	$d(f,g) = \max_{a \le x \le b} \|f(x) - g(x)\|$
$\mathscr{B}[a,b]$	$\|\| f \|\| = \sup_{a \le x \le b} \|f(x)\|$	$d(f,g) = \sup_{a \le x \le b} \|f(x) - g(x)\|$
$\mathscr{B}[\mathbb{Z}]$	$\|\| f \|\| = \sup_{a \le x \le b} \|f(x)\|$	$d(f,g) = \sup_{a \le x \le b} \|f(x) - g(x)\|$

Examples non-Banach spaces

$\mathscr{C}[a,b]$	$\|\| f \|\| = \left(\int_a^b \|f(x)\|^p\right)^{1/p}$	$d(f,g) = \left(\int_a^b \|f(x) - g(x)\|^p\right)^{1/p}$

Since $\mathbb{Q}^n$ over $\mathbb{R}$ is not a vector space, we cannot use this as an example of a non-Banach space even though it is an imcomplete metric space.

Proposition **p. 60** $\| \bullet \| : X \to \mathbb{R}$ is a continuous function.

Proof:
We only need show that $x_n \to x \Rightarrow \| x_n \| \to \| x \|$.
(Thm 1.4-8 Let $T : X \to Y$. (X, d), (Y, d') are metric spaces. T is continuous $\Leftrightarrow$ for every convergent sequence $x_n \to x$ in X, the sequence $Tx_n \to Tx$.)
Let $\varepsilon > 0$. If $x_n \to x$, then $\exists\, N \in \mathbb{N} \ni \| x_n - x \| < \varepsilon$.
$\| x_n \| = \| x_n - x + x \| \le \| x_n - x \| + \| x \|$.
$\| x \| = \| x - x_n + x_n \| \le \| x - x_n \| + \| x_n \|$.
Thus, $| \, \| x_n \| - \| x \| \, | \le \| x_n - x \| < \varepsilon$. $\therefore$ $\| x_n \| \to \| x \|$.

Remark This norm is Lipschitz continuous. (From Math 230b -

Definition $f : X \to Y$ is called Lipschitz if $\exists\, L \ge 0 \ni d(f(x), f(y)) \le Ld(x, y)$)
- We have regular continuity.
- We have uniform continuity.
- We have Lipschitz continuity, the strongest of all.

Remark $\| x_n \| \to \| x \|$ does not imply $x_n \to x$.
Proof:
Let $x_n \subset X = \{ z \in \mathbb{C} \mid \| z \| = 1\}$. Then $\| x_n \| = \| x \| = 1$ for all $x \in X$.
But x_n may not converge at all.

Theorem 2.3 - 1 A subspace of a Banach space is complete $\Leftrightarrow$ it is closed.

Definition Let (x_n) be a sequence in a normed space, X.
(x_n) *converges* if $\exists\, x \in X \ni \lim_{n \to \infty} \| x_n - x \| = 0$.

Definition We define the series $\sum_{n=1}^{\infty} x_n$ as the pair $((x_n), (s_k))$ where
$s_k = \sum_{n=1}^{k} x_n$, the sequence of partial sums.
$\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums converges,
$s_k \to s$.
We denote $\sum_{n=1}^{\infty} x_n = s$.

Remark If $\sum_{n=1}^{\infty} x_n$ converges, then $x_n \to \Theta$ (Rudin Theorem 2.23).
$x_n = s_n - s_{n-1} \to \Theta$ as (x_n) is Cauchy (Kreyszig, Thm 1.4-5 Every convergent sequence in a metric space is Cauchy.)

Definition $\sum_{n=1}^{\infty} x_n$ *converges absolutely* if $\sum_{n=1}^{\infty} \| x_n \|$ converges.

Note If $(x_n) \subset X$, a Banach space and $\sum_{n=1}^{\infty} \|x_n\|$ converges, then (s_k) is a Cauchy sequence.

Proof:

Let $\bar{s}_k = \sum_{n=1}^{k} \|x_n\|$.

$\sum_{n=1}^{\infty} \|x_n\|$ converges $\Rightarrow \| x_n \| \to 0$. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$.

Then $\exists\, K \in \mathbb{N} \ni \forall\, k \geq K$, $\| x_k \| < \varepsilon/n$. And we have

$$\begin{aligned} \| s_{n+k} - s_k \| &= \| x_{k+1} + x_{k+2} + \cdots + x_{n+k} \| \\ &\leq \| x_{k+1} \| + \| x_{k+2} \| + \cdots + \| x_{n+k} \| \\ &= \bar{s}_{n+k} - \bar{s}_k \\ &< \varepsilon. \end{aligned}$$

Thus, $\| s_{n+k} - s_k \| \to 0$.

Proposition If $(x_n) \subset X$ is a Banach space and $\sum_{n=1}^{\infty} \|x_n\|$ converges, then (s_k) converges.

Remark We know every convergent sequence is Cauchy. But in a Banach space, we can prove the inverse, every Cauchy sequence converges.

Proof:

We start by showing that if (x_n) is Cauchy, then $\| x_n \|$ is Cauchy.

Let $\varepsilon > 0$. Then $\exists\, N \in \mathbb{N} \ni \forall\, m, n \geq N$, $\| x_n - x_m \| < \varepsilon$.

As shown above, $| \, \| x_n \| - \| x_m \| \, | \leq \| x_n - x_m \| < \varepsilon$. $\therefore$ $\| x_n \|$ is Cauchy.

Let (x_n) be Cauchy in X. $\forall\, k \in \mathbb{N}\ \exists\, n_k \ni \|x_n - x_m\| < \dfrac{1}{2^k}$, $\forall\, n, m \geq n_k$.

Choose (n_k) to be an increasing sequence.

Let $y_1 = x_{n_1}$, $y_2 = x_{n_2} - x_{n_1}$, ..., $y_m = x_{n_m} - x_{n_{m-1}}$, ...

We want to show $\sum_{n=1}^{\infty} \|y_n\|$ is convergent.

We have that $\|x_{n_{k+1}} - x_{n_k}\| < \dfrac{1}{2^k}$ so $\sum_{n=1}^{\infty} \|y_n\| = \sum_{n=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\|$ converges.

Note that $s_k = \sum_{n=1}^{k} y_n = x_{n_1} + x_{n_2} - x_{n_1} + \cdots + x_{n_k} + x_{n_{k-1}} = x_{n_k}$.

So then $\sum_{n=1}^{\infty} \left(x_{n_{k+1}} - x_{n_k} \right)$ converges too.

Thus, $s_k = x_{n_k} \to x$. But now, $x_n \to x$ too.

Definition A set $\{e_1, e_2, ...\}$ forms a *Schauder basis* if $\forall\, x \in X$, $\forall\, \varepsilon > 0$, $\exists\, \alpha_1, \alpha_2, ..., \alpha_n \subset \mathbb{R} \ni \|x - \alpha e_i\| < \varepsilon$.

Example Let $f(x) = \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$ on $[a, b]$.
Every function in l^2 can be written in this way.
cos *nx*, sin *nx* are elements of the Schauder basis.

In l^2, $e_1 = (1, 0, 0, ...)$
$e_2 = (0, 1, 0, ...)$
$\cdots$

$x_n = 1/n$ cannot be written in terms of e_i.
$\forall\, x \in X\, \exists\, (\alpha_n) \subset \mathbb{R} \ni x = \sum_{n=1}^{\infty} \alpha_n e_n$.

Notes For Midterm 1,
know definitions and examples for
metric
subspace
open sets
closed sets
accumulation point
dense
complete metric space
vector space
normed space
Banach space
Schauder basis
Be able to prove theorem 1.3-4.
Be able to use, not to prove, Young's, Holder's, Minkowski's inequalities.
Know Lemma 2.2-9.
Every normed space is a metric space.
Continuity of a normed series (p. 60)